A Scientific Approach to CAPM and Options Valuation

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My Discomfort with CAPM

• Neoclassical finance:
  - Black Scholes derivation depends on replication (elegant)
  - CAPM depends on mean variance optimization (unrealistic)

• Would prefer one approach to both that starts from one logical principle I can accept
General Principle of Valuation

- Finance asks: What return should you expect from taking on risk?

- I claim: 
  Equal Risk Should Expect Equal Return $\mu$

- But some risks can be avoided

- Equal Unavoidable Risk Should Expect Equal Return $\mu$

- We know what expected return is, but what is risk?
How Can You Avoid Risk?

• Dilution:
  Combine security with a riskless bond

• Diversification:
  Combine security with many uncorrelated securities

• Hedging:
  Combine security with a correlated security
Specialize: Define Risk as Std Deviation

- Pretend that risk is the standard deviation $\sigma$ of returns

- Expected return $\mu$

- This is obviously naive. Of course there are much more general and realistic definitions of risk, but for now we stick with this one....
Equal Diluted Risk, Equal Return: All stocks have same Sharpe Ratio

• Combine weight $w$ of a risky stock $S(\mu, \sigma)$ with a weight $(1 - w)$ of riskless bond $B(r, 0)$

$$\begin{align*}
[w\mu + (1-w)r, w\sigma] &= [r + w(\mu - r), w\sigma] \\
\end{align*}$$

• All uncorrelated stocks with risk $w\sigma$ earn excess return $w(\mu - r)$

• One parameter fixes everything

$$\frac{\mu - r}{\sigma} = \lambda$$

• Same Sharpe ratio for all stocks
Equal Diversified Diluted Risk, Equal Return: Sharpe Ratio is Zero

• Suppose there are countless uncorrelated stocks \((\mu_i, \sigma_i)\)

• Put them all in a portfolio with weights \(P = \sum_i w_i S_i\)

• Then

\[
\mu - r \equiv \sum_i w_i (\mu_i - r) = \sum_i \lambda w_i \sigma_i
\]

• But the total risk \(\sigma\) diversifies to zero.

\[
\mu - r = \lambda \sigma = \lambda \sum_i \sigma_i = 0
\]

• Thus \(\lambda = 0\)

• Every stock is expected to earn the riskless rate
Equal Diversified Hedged Diluted Risk, Equal Return: CAPM

- Suppose there are countless stocks $S_i \ (\mu_i, \sigma_i)$ correlated with the market $M \ (\mu_M, \sigma_M)$

- Then the market-neutral stock $S_i^M \equiv S_i - \beta_i \left( \frac{|S_i|}{|M|} \right) M$

  has no market risk (is hedged), where $\beta_i = \rho_{im} \frac{\sigma_i}{\sigma_M}$

- Thus $S_i^M$ has zero Sharpe ratio, earns riskless rate $r$

- Which means

  $$(\mu_i - r) = \beta_i (\mu_M - r)$$
Stock and Option Have Equal Sharpe Ratio: Black-Scholes

- Stock S, call C

\[
\frac{(\mu_s - r)}{\sigma_s} = \frac{(\mu_c - r)}{\sigma_c}
\]

- Ito’s Lemma applied to a call C to obtains its expected return and effective volatility leads to Black-Scholes, as originally derived by Black.

- Unified treatment of BS and CAPM