Lecture 12: Jump Diffusion Models of the Smile.

12.1 Jumps

Why are we interested in jump models? Mostly, because of reality: stocks and indexes don't diffuse smoothly, and do seem to jump. Even currencies sometimes jump.

As an explanation of the volatility smile, jumps are attractive because they provide an easy way to produce the steep short-term skew that persists in equity index markets, and that indeed appeared soon after the jump/crash of 1987. Towards the end of this section we'll discuss the qualitative features of the smile that appears in jump models.

But jumps are unattractive from a theoretical point of view[1] because you cannot continuously hedge a distribution of finite-size jumps, and so risk-neutral arbitrage-free pricing isn't possible. As a result, most jump-diffusion models simply *assume* risk-neutral pricing without a thorough justification. It may make sense to think of the implied volatility skew in jump models as simply representing what sellers of options will charge to provide protection on an actuarial basis.

Whatever the case, there have been and will be jumps in asset prices, and even if you can't hedge them, we are still interested in seeing what sort of skew they produce.

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[1] "So what?" you may say.

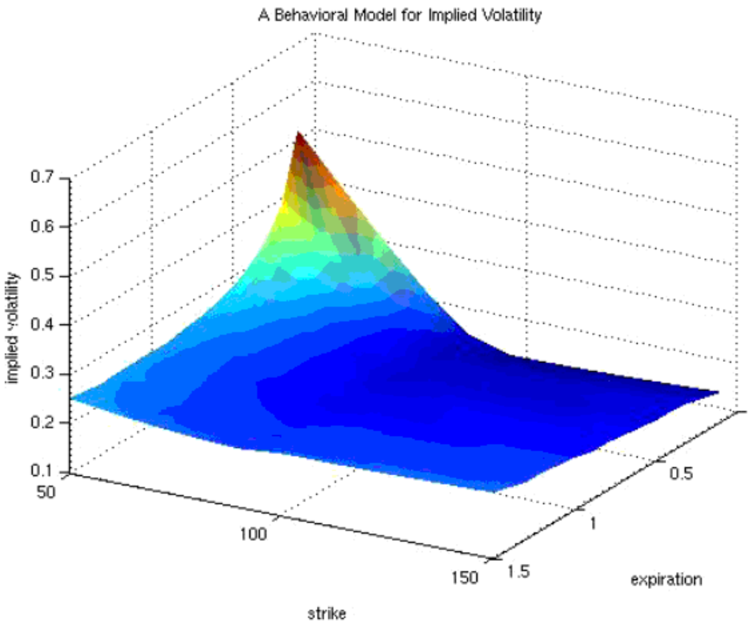
12.1.1 An Expectations View of the Skew Arising from Jumps

Assume that there is some probability *p* that a single jump will occur taking the market from S to K sometime before option expiration T, and that without that jump the future diffusion volatility of the index would have been $\sigma(T)$. Then the expected net future realized volatility $\sigma(S, K, T)$ contingent on the market jumping to strike K via a jump and a diffusion is approximately

$$T\sigma^2(S, K, T) = p \times \left[\frac{(S-K)}{S}\right]^2 + (1-p) \times T\sigma^2(T)$$

If implied volatility is expected future realized volatility, then $\sigma(S, K, T)$ is also the rational value for the implied volatility of an option with strike K. Below is the implied surface resulting from this picture. It’s not unrealistic for index options, especially for short expirations, and can be made more realistic by allowing the diffusion volatility $\sigma(T)$ to incorporate a term structure as well.

We choose *p* to be larger for a downward jump than for an upward jump.

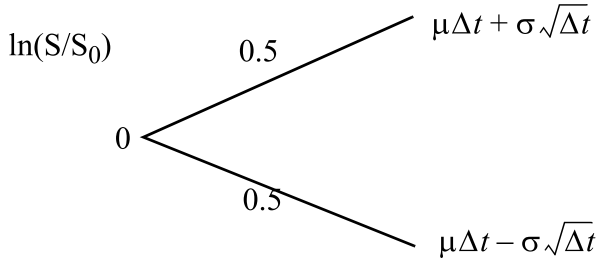


12.2 Modeling Jumps Alone

12.2.1 Stocks that Jump: Calibration and Compensation

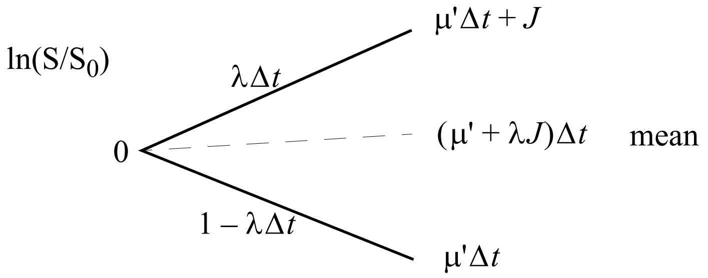
We've spent most of the course modeling pure diffusion processes. Now we'll look at pure jump processes as a preamble to examining the more realistic mixture of jumps and diffusion.

Here is a discrete binomial approximation to a diffusion process over time Δt:



The probabilities of both up and down moves are finite, but the moves themselves are small, of order $\sqrt{\Delta t}$. The net variance is $\sigma^2 \Delta t$ and the drift is μ. In continuous time this represents the process $d\ln S = \mu dt + \sigma dZ$.

Jumps are fundamentally different. There the probability of a jump J is small, of order Δt, but the jump itself is finite.



What does this process represent? Let's look at the mean and variance of the process.

$$
\begin{aligned}
E[\ln S] &= \lambda\Delta t[\mu'\Delta t + J] + (1 - \lambda\Delta t)\mu'\Delta t \\
&= (\mu' + \lambda J)\Delta t
\end{aligned}
$$

The variance of the process is given by

$$\begin{aligned} \text{var} &= \lambda\Delta t[J(1-\lambda\Delta t)]^2 + (1-\lambda\Delta t)[J\lambda\Delta t]^2 \\ &= (1-\lambda\Delta t)J^2\lambda\Delta t[1-\lambda\Delta t+\lambda\Delta t] \\ &= (1-\lambda\Delta t)J^2\lambda\Delta t \\ &\to J^2\lambda\Delta t \qquad \text{as } \Delta t \to 0 \end{aligned}$$

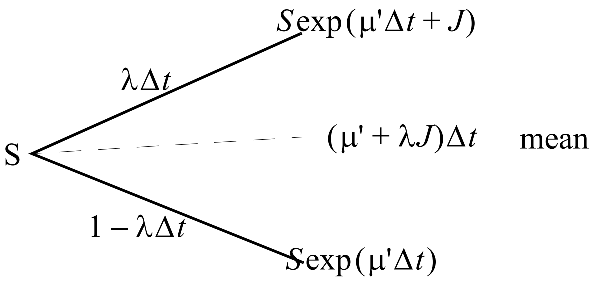
Thus, this process has an *observed* drift $\mu = (\mu' + \lambda J)$ and an observed volatility $\sigma = J^2\lambda$ .If we *observe* a drift μ and a volatility σ, and we want to obtain them from a jump process, we must calibrate the jump process so that

$$J = \frac{\sigma}{\sqrt{\lambda}}$$

$$\mu' = \mu - \sqrt{\lambda}\sigma$$

The one unknown is λ which is the probability of a jump in return of J in $\ln S$ per unit time.

This described how $\ln(S)$ evolves. How does S evolve?



$$\begin{aligned} E[S] &= \lambda\Delta t S\exp(\mu'\Delta t + J) + (1-\lambda\Delta t)S\exp(\mu'\Delta t) \\ &= S\exp(\mu'\Delta t)[1+\lambda(e^J-1)\Delta t] \\ &\approx S\exp[\{\mu' + \lambda(e^J-1)\}\Delta t] \end{aligned}$$

Thus if the stock grows risk-neutrally, for example, then

$$r = \mu' + \lambda(e^J - 1)$$

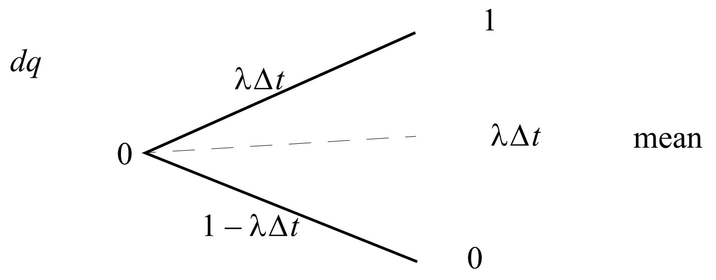
$$\mu' = r - (e^J - 1)$$

We have to c*ompensate the drift for the jump contribution to calibrate to a total return r.*

In continuous-time notation the elements of the jump can be written as a Poisson process

$$d\ln S = \mu' dt + J dq$$

Here dq is a jump or Poisson process that is modeled as follows:



The increment dq takes the value 1 with probability λdt if a jump occurs and the value 0 with probability $1 - \lambda dt$ if no jump occurs, so that the expected value $E[dq] = \lambda dt$.

12.2.2 The Poisson Distribution of Jumps

Let λ be the constant probability of a jump *J* occurring per unit time. Let $P(n, t)$ be the probability of *n* jumps occurring during time *t*. The probability of no jumps occurring during time *t* is given by the limit as $dt \to 0$ of the binomial Poisson process we wrote down above, so that

$$P[0, t] = (1 - \lambda dt)^{\frac{t}{dt}} = \left(1 - \lambda t\frac{dt}{t}\right)^{\frac{t}{dt}} = \left(1 - \frac{\lambda t}{N}\right)^{N} \to e^{-\lambda t} \text{ as } N \to \infty$$

where we wrote $N = t/(dt)$. Similarly

$$\begin{aligned} P(n, t) &= \frac{N!}{n!(N-n)!}(\lambda dt)^{n}(1 - \lambda dt)^{N-n} \\ &= \frac{N!}{n!(N-n)!}\left(\frac{\lambda t}{N}\right)^{n}\left(1 - \frac{\lambda t}{N}\right)^{N-n} \\ &= \frac{N!}{N^{n}(N-n)!}\frac{(\lambda t)^{n}}{n!}\left(1 - \frac{\lambda t}{N}\right)^{N-n} \\ &\to \frac{(\lambda t)^{n}}{n!}e^{-\lambda t} \end{aligned}$$

as $N \to \infty$ for fixed n. Note that $\sum_{n=0}^{\infty} P(n, t) = 1$

One can easily show that the mean number of jumps during time *t* is λt, confirming that λ should be regarded as the probability per unit time of one jump. One can also show that the variance of the number of jumps during time *t* is also λt.

12.2.3 Pure jump risk-neutral option pricing

We can value a standard call option (assuming risk-neutrality, i.e. taking the value as the expected risk-neutral discounted value of its payoffs) for a pure jump model as follows. It is the sum of the expected payoff for all numbers of jumps from 0 to infinity during time to expiration τ:

$$C = e^{-r\tau} \sum_{n=0}^{\infty} max[Se^{\mu'\tau + nJ} - K, 0] \frac{(\lambda\tau)^n}{n!} e^{-\lambda t}$$

where $Se^{\mu'\tau + nJ}$ is the final stock price after *n* Poisson jumps, and the payoff of the call is multiplied by the probability of the jump occurring, and

$$\mu' = r - (e^J - 1)$$

12.3 Modeling Jumps plus Diffusion

12.3.1 Some comments

- You can replicate an option exactly by means of a position of stock and several other options if the underlying stock undergoes only a finite number of jumps of known size. But with an infinite number of possible jumps, you cannot replicate; you can only minimize the variance of the P&L.

- Merton's model of jump-diffusion regards jumps as "abnormal" market events that have to be superimposed upon "normal" diffusion. This is in philosophical contrast to Mandelbrot, and to Eugene Stanley and his econophysics collaborators, who regard a mixture of two models for the world as being contrived; ideally, a single model, rather than a "normal" and "abnormal" model, should explain all events. Variance-gamma models also provide a unified view of market moves in which all stock price movements are jumps of various sizes.

12.3.2 Merton's jump-diffusion model and its PDE

Merton combines Poisson jumps with geometric Brownian diffusion, as follows

$$\frac{dS}{S} = \mu dt + \sigma dZ + Jdq \qquad \text{Eq.12.1}$$

where

$$\begin{aligned} E[dq] &= \lambda dt \\ var[dq] &= \lambda dt \end{aligned}$$

J very much resembles a random dividend yield paid on the stock; when a jump occurs the stock jumps (up or down) by a factor J. Later we will model J as a normal random variable.

You can derive a partial differential equation for options valuation under this jump-diffusion process, as follows.

Let $C(S, t)$ be the value of the option. We construct the usual hedged portfolio

$$\pi = C - nS$$

by shorting *n* shares of the stock S.

Now

$$\Delta C = \left[C_t + \frac{1}{2}C_{SS}(\sigma S)^2\right]dt + C_S(\mu S dt + \sigma S dZ) + [C(S + JS, t) - C(S, t)]dq$$

and

$$ndS = nS(\mu dt + \sigma dZ + Jdq)$$

$$\begin{aligned}\Delta\pi &= \Delta C - n[\mu S dt + \sigma S dZ + JS dq] \\ &= \left[C_t + C_S \mu S + \frac{1}{2}C_{SS}(\sigma S)^2 - n\mu S\right]dt + (C_S - n)\sigma S dZ \\ &\quad + [C(S + JS, t) - C(S, t) - nSJ]dq\end{aligned}$$

We can choose n to cancel the diffusion part of the stock price, so that $n = C_S$. Then the change in the value of the hedged portfolio becomes

$$\Delta\pi = \left[C_t + \frac{1}{2}C_{SS}(\sigma S)^2\right]dt + [C(S + JS, t) - C(S, t) - C_S SJ]dq \qquad \text{Eq.12.2}$$

The partially hedged portfolio is still risky because of the possibility of jumps. Suppose that despite the risk of jumps, we expect to earn the riskless return on the hedged position (this would be true, for example, if jump risk were truly diversifiable). Then $E[\Delta\pi] = r\pi\Delta t$ and $E[dq] = \lambda\Delta t$.

Applying this to Equation 13.2 we obtain

$$C_t + \frac{1}{2}C_{SS}(\sigma S)^2 + E[C((1 + J)S, t) - C(S, t) - C_S SJ]\lambda = (C - SC_S)r$$

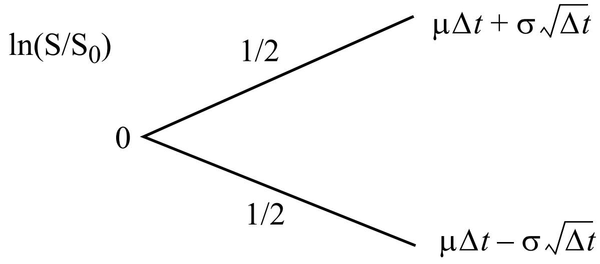
or

$$C_t + \frac{1}{2}C_{SS}(\sigma S)^2 + rSC_S - rC + E[C((1 + J)S, t) - C(S, t) - C_S SJ]\lambda = 0$$

where E[ ] denotes an expectation over jump sizes J. This is a mixed difference/partial-differential equation for a standard call with terminal payoff $C_T = max(S_T - K, 0)$. For $\lambda = 0$ it reduces to the Black-Scholes equation. We will solve it below by the Feynman-Kac method as an expected discounted value of the payoffs.

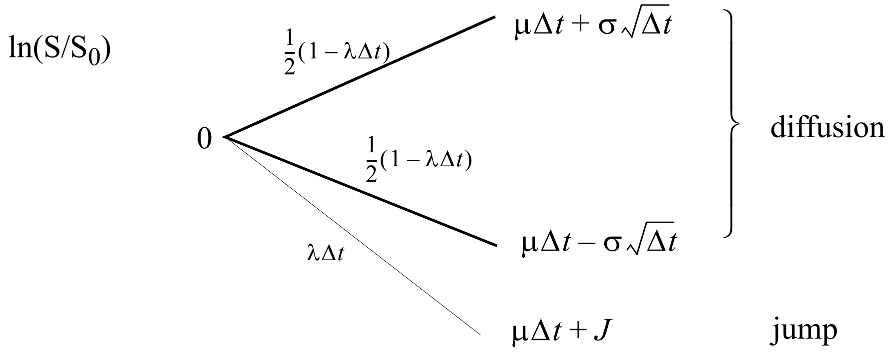
12.4 Trinomial Jump-Diffusion and Calibration

Diffusion can be modeled binomially, as in



The volatility σ of the log returns adds an Ito $\sigma^2/2$ term to the drift of the stock price S itself, so that for pure risk-neutral diffusion one must choose $\mu = r - \sigma^2/2$.

To add jumps one *J* needs a third, trinomial, leg in the tree:



Just as diffusion modifies the drift of the stock price, so do jumps.

12.4.1 The Compensated Process

How must we choose/calibrate the diffusion and jumps so that the stock grows risk-neutrally, i.e. that $E[dS] = Srdt$?

First let's compute the stock growth rate under jump diffusion.

$$E\left[\frac{S}{S_0}\right] = \frac{(1-\lambda\Delta t)}{2}e^{\mu\Delta t + \sigma\sqrt{\Delta t}} + \frac{(1-\lambda\Delta t)}{2}e^{\mu\Delta t - \sigma\sqrt{\Delta t}} + \lambda\Delta t e^{\mu\Delta t + J}$$

$$= e^{\mu\Delta t}\left[\frac{(1-\lambda\Delta t)}{2}(e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}}) + \lambda\Delta t e^{J}\right]$$

One can show by expanding this to keep terms of order Δt that

$$E\left[\frac{S}{S_0}\right] = e^{\left\{\mu + \frac{\sigma^2}{2} + \lambda(e^J - 1)\right\}\Delta t} + \text{higher order terms}$$

so that, if we want the stock to grow risk-neutrally, we must set

$$r = \mu + \frac{\sigma^2}{2} + \lambda(e^J - 1) \qquad \text{Eq.12.3}$$

So, to achieve risk-neutral growth in Equation 13.1, we must set the drift of the diffusion process to

$$\mu_{JD} = r - \underbrace{\frac{\sigma^2}{2}}_{\text{diffusion compensation}} - \underbrace{\lambda(e^J - 1)}_{\text{jump compensation}}$$

We have to set the continuous diffusion drift lower to *compensate* for the effect of both the diffusion volatility and the jumps, since both the jumps and the diffusion modify the expected return and the volatility.

12.5 Valuing a Call in the Jump-Diffusion Model

The process we are considering is

$$\frac{dS}{S} = \mu dt + \sigma dZ + Jdq \qquad \text{Eq.12.4}$$

where

$$\begin{aligned} E[dq] &= \lambda dt \\ var[dq] &= \lambda dt \end{aligned}$$

where at first J is assumed to be a fixed jump size, but will later be generalized to a normal variable. In order to achieve risk-neutrality, we set

$$\mu = r - \frac{\sigma^2}{2} - \lambda(e^J - 1) \qquad \text{Eq.12.5}$$

The value of a standard call in this model is given by

$$C_{JD} = e^{-r\tau}E[(S_T - K, 0)] \qquad \text{Eq.12.6}$$

The risk-neutral terminal value of the stock price is given by

$$S_T = Se^{\mu\tau + Jq + \sigma\sqrt{\tau}Z} \qquad \text{Eq.12.7}$$

where μ is given by Equation 12.5.

Now, in Equation 12.6 we have to sum over all the final stock prices, which we can break down into those with 0, 1, ... *n* ... jumps plus the diffusion, where the probability of n jumps in time τ is $\frac{(\lambda\tau)^n}{n!}e^{-\lambda\tau}$

Thus,

$$C_{JD} = e^{-r\tau}\sum_{n=0}^{\infty}\frac{\lambda\tau^n}{n!}e^{-\lambda\tau}E[max(S_T^n - K, 0)] \qquad \text{Eq.12.8}$$

where S_T^n is the terminal lognormal distribution of the stock price that started with initial price S and underwent *n* jumps as well as the diffusion.

The expected value in the above equation is an expectation over a lognormal stock price that, after time τ, has undergone *n* jumps, and therefore is simply related to a Black-Scholes expectation with a jump-shifted distribution or different forward price. In the risk-neutral world of Equation 12.5, the expected return on a stock that started at an initial price *S* and suffered *n* jumps is

$$\mu_n = r - \frac{\sigma^2}{2} - \lambda(e^J - 1) + \frac{nJ}{\tau}$$

where the last term in the above equation adds the drift corresponding to *n* jumps to the standard compensated risk-neutral drift $r - \frac{\sigma^2}{2}$, which appears in the Black-Scholes formula via the terms $d_{1,2}$.

Thus, since S_T is lognormal with a shifted center moved by *n* jumps,

$$E[max(S_T^n - K, 0)] \equiv e^{r_n\tau} C_{BS}(S, K, \tau, \sigma, r_n) \qquad \text{Eq.12.9}$$

where $C_{BS}(S, K, \tau, \sigma, r_n)$ is the standard Black-Scholes formula for a call with strike *K* and volatility σ with the drift rate r_n given by

$$r_n \equiv \mu_n + \frac{\sigma^2}{2} = r - \lambda(e^J - 1) + \frac{nJ}{\tau} \qquad \text{Eq.12.10}$$

Equation 12.10 omits the $\sigma^2/2$ term because the Black-Scholes formula for a stock with volatility σ already includes the term $\sigma^2/2$ in the $N(d_{1,2})$ terms as part of the definition of C_{BS}.

Combining Equation 12.8 and Equation 12.9 we obtain

$$\begin{aligned} C_{JD} &= e^{-r\tau} \sum_{n=0}^{\infty} \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau} e^{r_n\tau} C_{BS}(S, K, \tau, \sigma, r_n) \\ &= e^{-r\tau} \sum_{n=0}^{\infty} \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau} e^{\left(r - \lambda(e^J - 1) + \frac{nJ}{\tau}\right)\tau} C_{BS}(S, K, \tau, \sigma, r_n) \qquad \text{Eq.12.11} \\ &= e^{-(\lambda e^J \tau)} \sum_{n=0}^{\infty} \frac{(\lambda e^J \tau)^n}{n!} C_{BS}\left(S, K, \tau, \sigma, r + \frac{nJ}{\tau} - \lambda(e^J - 1)\right) \end{aligned}$$

Writing $\bar{\lambda} = \lambda e^J$ as the "effective" probability of jumps, we obtain

$$C_{JD} = e^{-\bar{\lambda}\tau} \sum_{n=0}^{\infty} \frac{(\bar{\lambda}\tau)^n}{n!} C_{BS}\left(S, K, \tau, \sigma, r + \frac{nJ}{\tau} - \lambda(e^J - 1)\right) \qquad \text{Eq.12.12}$$

This is a mixing formula. The jump-diffusion price is a mixture of Black-Scholes options prices with compensated drifts. This is similar to the result we got for stochastic volatility models with zero correlation -- a mixing theorem -- but here we had to appeal to the diversification of jumps or actuarial pricing rather than perfect riskless hedging.

Until now we assumed just one jump size *J*. We can generalize, as Merton did, to a distribution of normal jumps in return. Suppose

$$\begin{aligned} E[J] &= \bar{J} \\ var[J] &= \sigma_J^2 \end{aligned} \qquad \text{Eq.12.13}$$

describes the normal jump distribution.

Then

$$E[e^J] = e^{\bar{J} + \frac{1}{2}\sigma_J^2} \qquad \text{Eq.12.14}$$

Incorporating the expectation over this distribution of jumps into Equation 12.12 has two effects: first, *J* gets replaced by $\bar{J} + \frac{1}{2}\sigma_J^2$, and second, the variance of the jump process adds to the variance of the entire distribution in the Black-Scholes formula, so that we must replace σ^2 by $\sigma^2 + \frac{n\sigma_J^2}{\tau}$ because *n* jumps adds $\frac{n\sigma_J^2}{\tau}$ amount of variance. (The division by τ is necessary because variance is defined in terms of geometric Brownian motion and grows with time, but the variance of normally distributed J is independent of time.)

The general formula is therefore:

$$C_{JD} = e^{-\bar{\lambda}\tau}\sum_{n=0}^{\infty}\frac{(\bar{\lambda}\tau)^n}{n!}C_{BS}\left(S, K, \tau, \sqrt{\sigma^2+\frac{n\sigma_J^2}{\tau}}, r+\frac{n\left(\bar{J}+\frac{1}{2}\sigma_J^2\right)}{\tau}-\lambda\left(e^{\bar{J}+\frac{1}{2}\sigma_J^2}-1\right)\right)$$

where $\bar{\lambda} = \lambda e^{\bar{J}+\frac{1}{2}\sigma_J^2}$

If $\bar{J} = -\frac{1}{2}\sigma_J^2$ so that $E[e^J] = 1$ and the jumps add no drift to the process, then we get the simple intuitive formula

$$C_{JD} = e^{-\lambda\tau}\sum_{n=0}^{\infty}\frac{(\lambda\tau)^n}{n!}C_{BS}\left(S, K, \tau, \sqrt{\sigma^2+\frac{n\sigma_J^2}{\tau}}, r\right)$$

in which we simply sum over an infinite number of Black-Scholes distributions, each with identical riskless drift but differeing volatility dependent on the number of jumps and their distribution.

12.6 The Jump-Diffusion Smile (Qualitatively)

Jump diffusion tends to produce a steep realistic very short-term smile in strike or delta, because the jump happens instantaneously and moves the stock price by a large amount. Recall that stochastic volatility models, in contrast, have difficulty producing a very steep short-term smile unless volatility of volatility is very large.

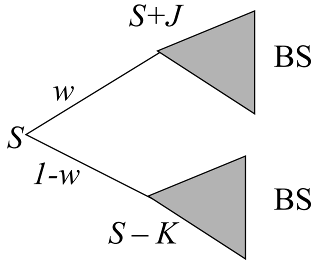
The long-term smile in a jump-diffusion model tends to be flat, because at large times the effect on the distribution of the diffusion of the stock price, whose variance grows like $\sigma^2\tau$, tends to overwhelm the diminishing Poisson probability of large moves via many jumps. Thus jumps produce steep short-term smiles and flat long-term smiles. Recall that mean-reverting stochastic volatility models also produce flat long-term smiles.

Jumps of a fixed size tend to produce multi-modal densities centered around the jump size. Jumps of a higher frequency tend to wash out the multi modal density and produce a smoother distribution of multiply overlaid jumps at longer expirations.

A higher jump frequency λ produces a steeper smile at expiration, because jumps are more probable and therefore are more likely to occur in the future as well.

Andersen and Andreasen claim that a jump-diffusion model can be fitted to the S&P 500 skew with a diffusion volatility of about 17.7%, a jump probability of $\lambda = 8.9\%$, an expected jump size of 45% and a variance of the jump size of 4.7%. A jump this size and with this probability seems excessive when compared to real markets, and suggests that the options market is paying a greater risk premium for protection against crashes.

12.7 An Intuitive Treatment of Jump Diffusion



Let's work out the consequences of a simple mixing model for jumps. You can think of the process as represented by the figure on the right, with J representing a big instantaneous jump up with a small probability *w*, and *K* representing a small move down with a large probability (*1-w*). Both moves are thereafter followed by diffusion with volatility σ. We assume that *w* is small and that it is initially sufficient to worry about only one jump, and ignore the effect of several jumps over the life of the option.

Risk-neutrality dictates that

$$S = w(S+J)+(1-w)(S-K) \qquad \text{Eq.12.15}$$

since the current stock price must be the discounted risk-neutral expected value of the stock prices at the next instant.

From this it follows that

$$K = \frac{w}{1-w}J \approx wJ$$

to leading order in *w*.

The mixing formula we derived for jump-diffusion dictates that the jump diffusion option price is given by

$$\begin{aligned} C_{JD} &= w \times C(S+J, \sigma)+(1-w) \times C(S-K, \sigma) \\ &\approx w \times C(S+J, \sigma)+(1-w) \times C(S-wJ, \sigma) \end{aligned} \qquad \text{Eq.12.16}$$

to leading order in *w*, where $C(S, \sigma)$ is our shorthand notation for the Black-Scholes option price for an option with strike K and volatility σ and time to expiration τ, but we have suppressed the explicit dependence on these variables which don't change here.

In writing the above formula, we have allowed for only zero or one jump in the mixing formula. Furthermore, we will makek approximations that work in the regime where the three dimensionless numbers w, $\sigma\sqrt{\tau}$ and J/S satisfy

$$w \ll \sigma\sqrt{\tau} \ll (J/S) \qquad \text{Eq.12.17}$$

so that the probability of a jump is much smaller than the square root of the stock's variance which itself is much smaller than the percentage jump size. In terms of real markets, we consider a small probability of a large jump, where a small probability means small relative to the square root of the variance of returns, and large jump means large relative to the same square root of the variance. We will then make approximations that keep only the leading order in w.

Let's now look at Equation 12.16 under these conditions, and stay close to the at-money strike.

Since $J/S \gg \sigma\sqrt{\tau}$, the positive jump J takes the call deep into the money, so that the first call $C(S+J, \sigma)$ in Equation 12.16 becomes equal to a forward whose value is

$$C(S+J, \sigma) \to (S+J) - Ke^{-r\tau}$$

For simplicity from now on we will also assume that $r = 0$ and ignore the effects of interest rates.

Under these circumstances,

$$\begin{aligned} C_{JD} &\approx w \times \{(S+J) - K\} + (1-w) \times C(S - wJ, \sigma) \\ &\approx w \times (J + S - K) + (1-w) \times \left\{ C(S, \sigma) - \frac{\partial C}{\partial S} wJ \right\} \end{aligned} \qquad \text{Eq.12.18}$$

We want to keep only terms of order *wS*, nothing smaller. For approximately at-the-money options, $C(S, \sigma) \sim S\sigma\sqrt{\tau}$, so we will neglect the term wC in the above equation since it is of order $w\sigma\sqrt{\tau}S$ which is smaller than wS.

Therefore Equation 12.18 becomes

$$\begin{aligned} C_{JD} &\approx C(S, \sigma) + w \times \left(J + S - K - \frac{\partial C}{\partial S} J \right) \\ &\approx C(S, \sigma) + w \times [J + S - K - N(d_1)J] \\ &\approx C(S, \sigma) + w \times [S - K + J\{1 - N(d_1)\}] \end{aligned}$$

Now close to at-the-money, we know that $N(d_1) \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\frac{\ln S/K}{\sigma\sqrt{\tau}}$

Therefore

$$C_{JD} \approx C(S, \sigma) + w \times \left[S - K + J\left\{\frac{1}{2} - \left[\frac{1}{\sqrt{2\pi}}\frac{\ln S/K}{\sigma\sqrt{\tau}}\right]\right\}\right]$$

Now close to at-the-money, the $S - K$ term in the above equation is negligible compared with J and the $J\ln S/K$ if $\sigma\sqrt{\tau}$ is small, because

$$J\frac{\ln S/K}{\sigma\sqrt{\tau}} = J\frac{\ln\left(1 + \frac{S-K}{K}\right)}{\sigma\sqrt{\tau}} \approx \frac{J}{K}\left\{\frac{S-K}{(\sigma\sqrt{\tau})}\right\} \approx \frac{S-K}{(\sigma\sqrt{\tau})} \gg S - K$$

Therefore

$$C_{JD} \approx C(S, \sigma) + wJ\left[\frac{1}{2} - \frac{1}{\sqrt{2\pi}}\frac{\ln S/K}{\sigma\sqrt{\tau}}\right] \qquad \text{Eq.12.19}$$

This is the approximate formula for the jump-diffusion call price in the case where we consider only one jump under the conditions $w \ll \sigma\sqrt{\tau} \ll (J/S)$, i.e. a *small* probability (relative to volatility) of a *large* one-sided jump. Recall that $C(S, \sigma)$ is the Black-Scholes option price.

Someone using the Black-Scholes model to interpret a jump-diffusion price will quote the price as $C(S, \Sigma)$ where Σ is the implied volatility smile function. We can write

$$C(S, \Sigma) = C(S, \sigma + \Sigma - \sigma) \approx C(S, \sigma) + \frac{\partial C}{\partial \sigma}(\Sigma - \sigma) \qquad \text{Eq.12.20}$$

Comparing Equation 12.20 and Equation 12.19 we obtain

$$\Sigma \approx \sigma + \frac{wJ\left[\frac{1}{2} - \frac{1}{\sqrt{2\pi}}\frac{\ln S/K}{\sigma\sqrt{\tau}}\right]}{\frac{\partial C}{\partial \sigma}}$$

For options close to at the money, $\frac{\partial C}{\partial \sigma} = S\sqrt{\tau}N'(d_1) \approx \frac{S\sqrt{\tau}}{\sqrt{2\pi}}$, so that

$$\Sigma \approx \sigma + \frac{wJ}{S\sqrt{\tau}}\left[\sqrt{\frac{\pi}{2}} + \frac{\ln K/S}{\sigma\sqrt{\tau}}\right] \qquad \text{Eq.12.21}$$

We see that the jump-diffusion smile is linear in $\ln S/K$ when the strike is close to being at-the money, and the implied volatility increases when the strike increases, as we would have expected for a positive jump *J*.

We can examine this a little more closely for small and large expirations. In the Merton model we showed that $w = \bar{\lambda}\tau e^{-\bar{\lambda}\tau}$ where $\bar{\lambda} = \lambda e^{J}$ and λ was the probability of a jump per unit time. Inserting this expression for *w* into Equation 12.21 leads to

$$\begin{aligned}\Sigma &\approx \sigma + \bar{\lambda}\sqrt{\tau}e^{-\bar{\lambda}\tau}\frac{J}{S}\left[\sqrt{\frac{\pi}{2}} + \frac{\ln K/S}{\sigma\sqrt{\tau}}\right] \\ &= \sigma + \bar{\lambda}e^{-\bar{\lambda}\tau}\frac{J}{S}\left[\sqrt{\frac{\pi\tau}{2}} + \frac{\ln K/S}{\sigma}\right]\end{aligned} \qquad \text{Eq.12.22}$$

As $\tau \to 0$ for short expirations, the implied volatility smile becomes

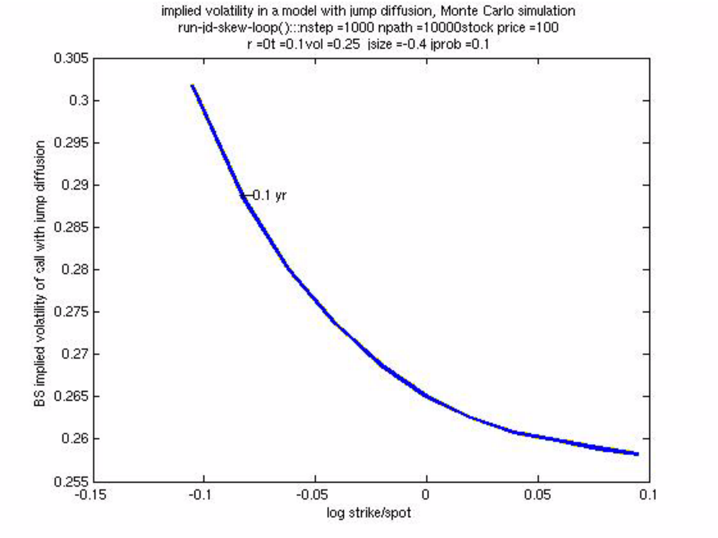
$$\Sigma = \sigma + \bar{\lambda}\frac{J}{S}\left[\frac{\ln K/S}{\sigma}\right],$$

a finite smile proportional to the percentage jump and its probability, and linear in $\ln\frac{K}{S}$. The greater the expected jump, the greater the skew. This is a model suitable for explaining the short-term equity index skew.

For long expirations the approximations required by are no longer valid.Nevertheless, the formula Equation 12.22 illustrates that, as $\tau \to \infty$, the exponential time decay factor $e^{-\bar{\lambda}\tau}$ drives the skew to zero. Asymmetric jumps produce a steep short-term skew and a flat long-term skew,

Here are two figures for the smile in a jump model, obtained from Monte Carlo simulation with a fixed jump size.

This is the skew for a jump probability of 0.1 and a percentage jump size of -0.4, with a diffusion volatility of 25%, for options with 0.1 years to expiration.



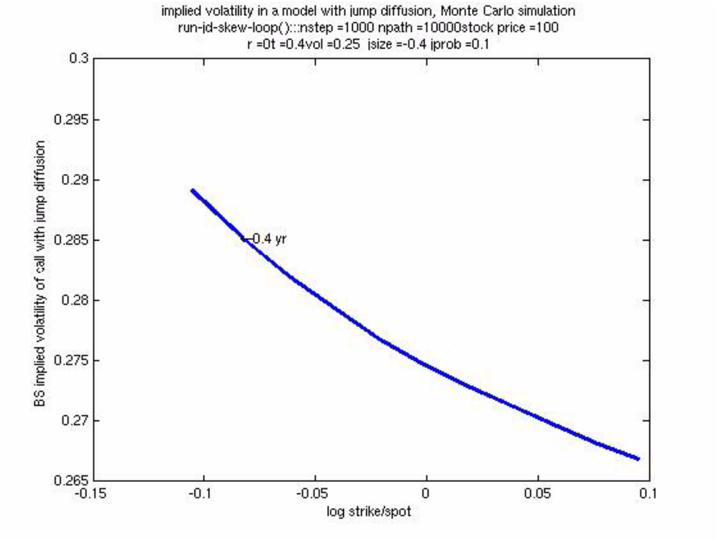
Equation 12.22 gives the approximate formula

$$\Sigma \approx \sigma + \bar{\lambda} e^{-\bar{\lambda}\tau} \frac{J}{S}\left[\sqrt{\frac{\pi\tau}{2}} + \frac{\ln K/S}{\sigma}\right]$$

$$\approx 0.266 + 0.16 \ln \frac{K}{S}$$

which matches the graph pretty well at the money.

Here is a similar skew for an option with 0.4 years to expiration.



It’s still not a bad fit, but because of the longer expiration our approximation of mixing between only zero and one instantaneous immediate jump is not as good. One would have to amend the approximation by allowing for jumps that occur throughout the life of the option.